Adrien-Marie Legendre (1752 – 1833) conjectured that a prime number exists between any two consecutive perfect squares. That is, given a natural number, n, there exist a prime between n^2 and $(n+1)^2$. Though it is an easy conjecture to make and it seems to be true, proving this conjecture has proven to be difficult. In 1912, Edmund Landau listed this conjecture along with four others in number theory as problems that were "unattackable at the present state of mathematics." I disagree.

This conjecture is not "unattackable," and its proof is not that difficult once we establish a special arrangement of the natural numbers.

Conjecture: Given a natural number, n, there exist a prime number between n^2 and $(n+1)^2$.

Proof: We can list the natural numbers from 1 to n, using set notation L = {1, 2, 3, … , n}. Now, there must be a finite number of odd primes within the set L. We can list those primes using set notation as follows: P = {3, 5, 7, …, p_i}.

We know that given an infinite set of natural numbers we can create infinite subsets defined by "r mod n," where r and n are natural numbers (i.e. r, n $\in \mathbb{N}$). We have the following closed intervals:

$$[1, 2, 3, \ldots, n]$$

$$[(n+1), (n+2), \ldots, (n+n)]$$

$$[(2n+1), (2n+2), \ldots, (2n+n)]$$

$$[(3n+1), (3n+2), \ldots, (3n+n)]$$

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Within these intervals, if we attempt to minimize the quantity of numbers relatively prime to the prime numbers in P, then we can find one such interval that leaves at most two numbers within the interval that are relatively prime to {3, 5, 7, …, p_i}. A method for finding this interval is described next.

We know that for any given natural number, n, that natural number is either even or odd. Without loss of generality (WLOG), let n be an odd number. We can multiply all the odd numbers less than or equal to n to create a natural number 'HF' = 3·5·7···O_i. Dividing this number 'HF' by 2 we have the following rational number: $\frac{3\cdot5\cdot7\cdots O_i}{2}$. We can create an interval of natural numbers of size (n+1) using ('HF' ÷ 2) as a midpoint. We have the following interval (I use l_i to represent unique factors for each number in the interval, so l_i is not the same number for any number in the interval … It is strictly a placeholder):

$$\left[n \cdot l_i,\ (n-2)\cdot l_i,\ (n-4)\cdot l_i, \dots,\ 7\cdot l_i,\ 5\cdot l_i,\ 3\cdot l_i,\ \left(\frac{'HF'}{2}-\frac{1}{2}\right),\ \left(\frac{'HF'}{2}+\frac{1}{2}\right),\ 3\cdot(l_i+1),\ 5\cdot(l_i+1),\ 7\cdot(l_i+1), \dots,\ (n-4)\cdot(l_i+1),\ (n-2)\cdot(l_i+1),\ n\cdot(l_i+1)\right]$$

Allow me to present an example of such an interval using the natural number 9. Using 9, we have HF_9 = (3·5·7·9) = 945. If we divide 945 by 2, we get 472.5. Thus, we have the following interval:

Row 1:	1	2	3	4	5	6	7	8	9	10
Row 2:	468	469	470	471	472	473	474	475	476	477
Row 3:	9	7	5	3	RP	RP	3	5	7	9
Row 4:	52	67	94	157	RP	RP	158	95	68	53

Table 1. *There are at least two relatively prime numbers in each interval*

In the above table, "Row 1" is simply an enumeration of each term in the interval. "Row 2" is the interval itself or the interval we are investigating. "Row 3" is a list of the odd numbers that divide the individual terms in the interval; notice how 472 and 473 are relatively prime (RP) to each odd number in the interval [1, …, 9]. "Row 4" is a list of the l_i s for each term; that is, 9·52=468, 7·67=469, and so on for each term. Do you understand how the first, general, interval now makes sense?

We can create such an interval for any odd natural number, n. This tells us, that taking only the odd primes into consideration, each such interval must have at least two numbers that are relatively prime to the odd primes less than n.

We have proven the first part of Legendre's conjecture (you'll see what I mean shortly). There are only two parts to this proof. The second part follows.

I will first show you how the proof works as a continuation of our example using the natural number 9. Then I'm going to state the proof in general.

We know that 945 is divisible by each number in {3, 5, 7, 9}. Observe what happens when we multiply each term in "Row 2" (Table 1) by 2, and then subtract half of the terms from 945 and from the other half of the terms we subtract 945. Essentially, "Row 3" represents the absolute value of difference between "Row 1" and "Row 2."

Row 1:	945	945	945	945	945	945	945	945	945	945
Row 2:	936	938	940	942	944	946	948	950	952	954
Row 3:	9	7	5	3	1	1	3	5	7	9

Table 2. *First row mapping to find position of odd numbers*

If we continue this process for numbers in Table 1 less than 468, we have the following sequence 467, 466, 465, and so on, which implies that we have an interval beginning with terms 934, 932, 930, and so on when those terms are multiplied by 2. Notice that subtracting these values from 945 we have (945 – 934), (945 – 932), and (945 – 930) ⇒ 11, 13, and 15. In this manner we can construct a table, with origin 472.5, that gives us an interval of numbers representing all odd natural numbers between 9 and 81 in ascending order. We have the following two tables:

Table 3. *Starting at 472, we write the natural numbers in descending order, down to 432. Notice that 459 is 9·51, 450 is 9·50, … Also notice that 472.5 – (81/2) = 432.*

432	433	434	435	436	437	438	439	440
441	442	443	444	445	446	447	448	449
450	451	452	453	454	455	456	457	458
459	460	461	462	463	464	465	466	467
468	469	470	471	472				

Table 4. *This table represents the numbers in Table 3 multiplied by 2. We use these values to subtract from 945 to find the location of odd numbers in [1, …, 81].*

864	866	868	870	872	874	876	878	880
882	884	886	888	890	892	894	896	898
900	902	904	906	908	910	912	914	916
918	920	922	924	926	928	930	932	934
936	938	940	942	944				

We can map the natural numbers in Table 4 to the origin of the natural number system, to produce all odd numbers from 1 to 81 (which is 9^2). But before we do that let us add another row to the top of the table to include the natural odd numbers between 81 and 100. We have the following implication:

846	848	850	852	854	856	858	860	862
864	866	868	870	872	874	876	878	880
882	884	886	888	890	892	894	896	898
900	902	904	906	908	910	912	914	916
918	920	922	924	926	928	930	932	934
936	938	940	942	944				

$\Rightarrow$

99	97	95	93	91	89	87	85	83
81	79	77	75	73	71	69	67	65
63	61	59	57	55	53	51	49	47
45	43	41	39	37	35	33	31	29
27	25	23	21	19	17	15	13	11
9	7	5	3	1				

We know from "Part One" of this proof that there must be at least two numbers in [846, 848, 850, … 862] that are relatively prime to the numbers in {3, 5, 7, 9}. This implies that there must be two numbers in [83, 85, 87, … 99] that are relatively prime to the numbers in {3, 5, 7, 9}. How do we know this? Because 945 is a product of each prime that can divide the odd numbers in [83, 99]. Subtracting any number in [846, 862] from 945 that is relatively prime to {3, 5, 7, 9} will result in a natural number that is relatively prime to {3, 5, 7, 9} in [83, 89]. And that implies that there are **at least two prime numbers** in [83, 85, 87, … 99]. In fact, we have three prime numbers in that interval, namely 83, 89, and 97!

In general, we are exploiting the following relationship of natural numbers:

$a \pm b = (p_i \cdot j \pm p_i \cdot k) = p_i \cdot (j \pm k)$,where a, b, j, k, p_i are natural numbers.

In our example, a = 945 and b = {846, 848, 850, 852, 854, 856, 858, 860, 862}. Therefore, any natural number in b that is relatively prime to the prime numbers in {3,5,7,9} gives us odd numbers in [83, 99] that are relatively prime to {3,5,7,9}. This must mean that those numbers, 83, 89, and 97, are prime numbers since they are relatively prime to the primes less than the square root of $(n+1)^2$ = 10 and are vacuously relatively prime to 2.

We can generalize this process (of table creation and mapping) for any odd number n at infinity. We simply find the 'HF' value and divide by 2. Then, by the well-ordering principle, we can find the greatest odd number, n = O_g, that is a factor of 'HF' and divide that odd natural number by 2. We obtain the following bounds for our interval (to create a table): $\left[\left(\frac{'HF'}{2} - \frac{O_g}{2}\right), \left(\frac{'HF'}{2} - \frac{1}{2}\right)\right]$. We can extend this table one more row at the top to obtain the odd numbers between n^2 and $(n+1)^2$. *Notice, that if n is even then we can create a table using (n+1) and we would simply not add a top row to our table – we still have two natural numbers in the top row that are relatively prime to the primes in the set {3, 5, 7, …, n}.* Because there are always at least two odd numbers relatively prime to the odd numbers in {3, 5, 7, …, n} in this final added interval, we must have at least two prime numbers in that interval. So, Legendre was right, there is always one prime between two consecutive perfect squares. Further, there are actually always at least two primes in that interval.

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Proof by R. D. L'Felix (Nov. 2019)